Fractional Thermoelectric Viscoelastic Materials

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ABSTRACT: A new mathematical model of thermoelectro viscoelasticity theory was constructed in the context of a new consideration of heat conduction with fractional order. The state space approach developed earlier by Ezzat was adopted for the solution of a one-dimensional problem in the presence of heat sources. The Laplace-transform technique was used. A numerical method was employed for the inversion of the Laplace transforms. According to the numerical results and their graphs, a conclusion about the new theory was constructed. Some comparisons are shown in figures to estimate the effect of the fractional order parameter on all of the studied fields. © 2011 Wiley Periodicals, Inc. J Appl Polym Sci 124: 2187–2199, 2012

Key words: solid-state structure; strain; stress; thermal properties; viscoelastic properties

INTRODUCTION

The linear theory of elasticity is of paramount importance in the stress analysis of steel, which is the most common engineering structural material. To a lesser extent, linear elasticity describes the mechanical behavior of the other common solid materials, for example, concrete, wood, and coal. However, the theory does not apply to the behavior of many new synthetic materials of the elastomer and polymer type, such as poly(methyl methacrylate) (Perspex), polyethylene, and poly(vinyl chloride).

With the rapid development of polymer science and the plastics industry, as well as the widespread use of materials under high temperature in modern technology and the application of biology and geology in engineering, the theoretical study and applications of viscoelastic materials have become important tasks for solid mechanics.

Linear viscoelasticity remains an important area of research not only because of the advent and use of polymers but also because most solids, when subjected to dynamic loading, exhibit viscous effects.[1] The stress–strain law for many materials, including polycrystalline metals and high polymers, can be approximated by linear viscoelasticity theory.[2] The mechanical model representation of linear viscoelastic behavior results has been investigated by Gross,[3] Staverman and Schwarzl,[4] Alfery and Gurnee,[5] and Ferry.[6] One can refer to Ilioushin and Pobedria[7] for the formulation of a mathematical theory of thermal viscoelasticity and for the solutions of some boundary value problems and to Pobedria[8] for coupled problems in continuum mechanics.

The modification of the heat-conduction equation from diffusive to a wave type may be affected either by a microscopic consideration of the phenomenon of heat transport or in a phenomenological way by modification of the classical Fourier law of heat conduction. The first is due to Cattaneo,[9] who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier law. Lord and Shulman[10] introduced the theory of generalized thermoelasticity with one relaxation time (τ_0) for the special case of an isotropic body. This theory was extended by Sherief and Dhaliwal[11] to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and, hence, eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and coupled theories of thermoelasticity. Earlier, we investigated[12] the propagation of discontinuities of solutions in this theory.

The generalized thermoviscoelasticity models ignoring the relaxation effects of the volume, were established by Ezzat et al.[13] and Othman at el.[14] Among the theoretical contributions to the subject have been our proofs of uniqueness theorems under different conditions[15] and the boundary element formulation.[16] For a half-space of an electrically conducting viscoelastic

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material, a lot of problems describing interesting phenomena that characterize different theories of generalized thermoviscoelasticity have been solved by many researchers,[17–20] among others.

Differential equations of fractional order have been the focus of many studies because of their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. The most important advantage of the use of fractional differential equations in these and other applications is their nonlocal properties. It is well known that the integer-order differential operator is a local operator but the fractional-order differential operator is nonlocal. This means that the next state of a system depends not only on its current state but also on all of its historical states. This is more realistic, and it is one reason that fractional calculus has become more and more popular (see Caputo,[21] Podlubny,[22] and Mainardi and Gorenflo[23]).

Although the tools of fractional calculus have been available and are applicable to various fields of study, the investigation of the theory of fractional differential equations was started quite recently by Caputo.[21] Differential equations involving Riemann–Liouville differential operators of the fractional order $0 < \alpha < 1$, appeared to be important in the modeling of several physical phenomena in Kiryakova[24] and, therefore, seem to deserve independent study of their theory parallel to the well-known theory of ordinary differential equations.

Fractional calculus has been used successfully to modify many existing models of physical processes. One can state that the whole theory of fractional derivatives and integrals was established in the second half of the 19th century. The first application of fractional derivatives was given by Abel,[25] who applied fractional calculus to the solution of an integral equation that arose in the formulation of the Tautochrone problem.[26] The generalization of the concept of derivatives and integrals to a noninteger order has been subjected to several approaches, and some various alternative definitions of fractional derivatives have appeared in Oldham and Spanier,[27] Miller and Ross,[28] Samko et al.,[29] Gorenflo and Mainardi,[30] and Hilfer.[31] In the last few years, fractional calculus has been applied successfully in various areas to modify many existing models of physical processes, for example, in chemistry, biology, modeling and identification, electronics, wave propagation, and viscoelasticity in Caputo and Mainardi,[32] Caputo,[33] Bagley and Torvik,[34] Koeller,[35] and Rossikhin and Shitikova.[36] One can refer to Podlubny[22] for a survey of applications of fractional calculus.

Recently, a considerable research effort was expended to study anomalous diffusion, which was characterized by the time-fractional diffusion-wave equation by Kimmich:[37]

$$\rho C = \lambda I^{\upsilon} \nabla^2 C, 0 < \upsilon \leq 2 \tag{1}$$

where ρ is the density, C is the concentration, λ is the diffusion conductivity, and I^{υ} is the Riemann–Liouville fractional integral.

I^{υ} was introduced as a natural generalization of the well-known n-fold repeated integral $I^n f(t)$, written in a convolution-type form by Mainardi and Gorenflo:[23]

$$\left.\begin{aligned} I^{\upsilon} f(t) &= \tfrac{1}{\Gamma(\upsilon)} \int_0^t (t-\xi)^{\upsilon-1} f(\xi)\, d\xi \\ I^0 f(t) &= f(t) \end{aligned}\right\} 0 < \upsilon \leq 2 \tag{2}$$

where Γ is the gamma function and t is the time. According to Kimmich,[37] Eq. (1) describes different cases of diffusion where $0 < \upsilon < 1$ corresponds to weak diffusion (subdiffusion), $\upsilon = 1$ corresponds to normal diffusion, $0 < \upsilon < 2$ corresponds to strong diffusion (superdiffusion), and $\upsilon = 2$ corresponds to ballistic diffusion. It should be noted that the term *diffusion* is often used in a more generalized sense, including in various transport phenomena. Equation (1) is a mathematical model of a wide range of important physical phenomena, for example, the subdiffusive transport that occurs in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous, and random media. Superdiffusion is comparatively rare and has been observed in porous glasses, polymer chains, and biological systems and is the transport of organic molecules and atomic clusters on the surface. One might expect anomalous heat conduction in media where anomalous diffusion is observed.

Fujita[38] considered the constitutive equation for the heat flux in the following form:

$$\mathbf{q} = -\kappa I^{\upsilon-1} \nabla T, 1 \leq \upsilon \leq 2 \tag{3}$$

where $\mathbf{q}$ is the heat flux vector, κ is the thermal conductivity, and ∇T is change in the absolute temperature.

Povstenko[39] used the Caputo heat-wave equation to define the fractional heat-conduction equation in the following form:

$$\mathbf{q} = -\kappa I^{\upsilon-1} \nabla T, 0 < \upsilon \leq 2 \tag{4}$$

Cattaneo[9] introduced a law of heat conduction to replace the classical Fourier law in the following form:

$$\mathbf{q} + \tau_0 \frac{\partial \mathbf{q}}{\partial t} = -\kappa \nabla T \tag{5}$$

Sherief et al.[40] introduced a formula of heat conduction and took into account Eq. (7):

$$\mathbf{q} + \tau_0 \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -\kappa \nabla T, 0 < \alpha \leq 1 \tag{6}$$

where

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = \begin{cases} f(x,t) - f(x,0) & \alpha \to 0 \\ I^{\alpha-1} \frac{\partial f(x,t)}{\partial t} & 0 < \alpha < 1 \\ \frac{\partial f(x,t)}{\partial t} & a = 1 \end{cases} \tag{7}$$

They proved a uniqueness theorem and derived a reciprocity relation and a variational principle.

In the limit, as α moves toward 1, Eq. (6) is reduced to the well-known Cattaneo law used by Lord and Shulman[10] to derive the equation of the generalized theory of thermoelasticity with one relaxation time. It is known that Lebon et al.[41] and Jou et al.[42] showed that the classical entropy derived with this law, instead of being monotonically increasing, behaves in an oscillatory way. Strictly speaking, this result is not incompatible with the Clausius' formulation of the second law, which states that the entropy of the final equilibrium state must be higher than the entropy of the initial equilibrium state. However, the nonmonotonic behavior of the entropy is in contradiction with the local equilibrium formulation of the second law, which requires that the entropy production must be positive everywhere at any time, as Lebon et al.[41] During the last 2 decades, this has become the subject of many research articles and has resulted in the introduction of what is known now as *extended irreversible thermodynamics*. A review can be found in Jou et al.[42]

Youssef[43] introduced another formula of heat conduction that took into consideration eqs. (3)–(5):

$$\mathbf{q} + \tau_0 \frac{\partial \mathbf{q}}{\partial t} = -\kappa I^{\upsilon-1} \nabla T, 0 < \upsilon \leq 2 \tag{8}$$

A uniqueness has been proven.

We[44] introduced two general models of a fractional heat-conduction law for nonhomogenous anisotropic elastic solids. Uniqueness and reciprocal theorems were proven, and the convolutional variational principle was established and used to prove a uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors, except symmetry conditions. The two-temperature dynamic coupled theory (Lord–Shulman) and fractional coupled thermoelasticity theory resulted as limit cases. For fractional thermoelasticity, which does not involve two temperatures, we[45] established the uniqueness theorem, reciprocal theorems, and convolution principle. The dynamic coupled and Green–Naghdi thermoelasticity theories resulted as limit cases. The reciprocity relation, in the case of a quiescent initial state, was found to be independent of the order of differintegration.[44,45]

Earlier, we[46,47] investigated the fractional order theory of a perfect conducting thermoelastic medium and the theory of fractional order in electrothermoelasticity.

Thermoelectric devices have many attractive features, such as a long life, no moving parts, no noise, easy maintenance, and high reliability, compared with conventional fluid-based refrigerators and power-generation technologies. However, their use has been limited by the relatively low performance of current thermoelectric materials. The efficiency of a thermoelectric material is related to the so-called dimensionless thermoelectric figure of merit (ZT). ZT was defined in Goldsmid:[48]

$$ZT = \frac{\sigma_0 S^2}{\kappa} T \tag{9}$$

where σ_0 is the electric conductivity and S is the Seebeck coefficient.

The best thermoelectric materials that are currently in devices have a value of $ZT \gg 1$.

A related effect (the Peltier effect) was discovered a few years later by Peltier, who observed that when an electrical current is passed through the junction of two dissimilar materials, heat is either absorbed or rejected at the junction, depending on the direction of the current. This effect is due to the difference in the Fermi energies of the two materials. The absolute temperature (T), S, and Peltier coefficient (Π) are related by the first Thomson relation, as discussed by Morelli:[49]

$$\Pi = ST \tag{10}$$

In this study, a new model of the time-fractional derivative of α in the heat-conduction equation was derived in the context of generalized thermoelectric elasticity theory. The governing coupled equations were applied to a problem of an electroconducting half-space with heat source distribution in the presence of a transverse magnetic field. Laplace transforms and state space approach techniques (Ezzat[50]) were used to obtain the solution. Laplace transforms were obtained with the complex inversion formula of the transform, together with Fourier expansion techniques proposed by Honig and Hirdes.[51] The effects of various physical parameters on various stress and heat-transfer characteristics are discussed in detail and are represented graphically.

DERIVATION OF THE FRACTIONAL HEAT-CONDUCTION EQUATION

Conventional electrothermoelasticity is based on the principles of the classical theory of heat conductivity, specifically on the classical Fourier's law, which relates $\mathbf{q}$ and the conduction current–density vector ($\mathbf{J}$) to the temperature gradient, as given by[52]

$$\mathbf{q} = -\kappa\nabla T + \Pi\mathbf{J} \tag{11}$$

$$\mathbf{J} = \sigma_0\left(E + \frac{\partial\mathbf{u}}{\partial t}\wedge B - S\nabla T\right) \tag{12}$$

where E is the total electric field, $\mathbf{u}$ is the displacement vector, and $\mathbf{B}$ is the magnetic induction vector. The energy equation in terms of $\mathbf{q}$ introduced by Biot[53] is

$$\frac{\partial}{\partial t}(\rho C_E T + \gamma T_o \mathbf{e}) = -\nabla\mathbf{q} + Q, \tag{13}$$

where C_E is the specific heat at a constant strain, $\gamma = (3\lambda + 2\mu)\alpha_t$ (where λ and μ are Lame's constants and αt is the coefficient of linear thermal expansion), T_0 is the reference temperature, $\mathbf{e}$ is the strain deviator tensor, and Q is the intensity of the applied heat source per unit volume.

Over the past 3 decades, nonclassical electrothermoelasticity theories, in which Fourier law [Eq. (11)] and the heat equation [Eq. (13)] were replaced by more general equations, have been formulated with Taylor's series used to expand $\mathbf{q}(x,t + \tau_0)$ and with terms up to the first order in τ_0 retained. The first well-known generalization of such a type is as follows:[54]

$$\mathbf{q} + \tau_0\frac{\partial\mathbf{q}}{\partial t} = -\kappa\nabla T + \Pi\mathbf{J} \tag{14}$$

This generalization leads to the hyperbolic-type heat-transport equation in the theory of electrothermoelasticity:[55,56]

$$\frac{\partial}{\partial t}\left(1 + \tau_0\frac{\partial}{\partial t}\right)(\rho C_E T + \gamma T_0 e) = \kappa\nabla^2 T - \nabla\Pi\mathbf{J} + Q + \tau_0\frac{\partial Q}{\partial t} \tag{15}$$

In this study, the new fractional Taylor's series of time-fractional order α developed by Jumarie[57] was adopted to expand $\mathbf{q}(x,t + \tau_0)$, and with terms up to α in the thermal τ_0 retained, we obtained

$$\mathbf{q}(x, t + \tau_0) = \mathbf{q}(x, t) + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha\mathbf{q}}{\partial t^\alpha}, \quad 0 < \alpha \le 1. \tag{16}$$

From a mathematical viewpoint, Fourier law [Eq. (11)] in the theory of generalized fractional heat conduction is given by

$$\mathbf{q} + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha q}{\partial t^\alpha} = -\kappa\nabla T + \Pi\mathbf{J}, \quad 0 < \alpha \le 1 \tag{17}$$

Taking the partial time derivative of α of Eq. (13), we get[58]

$$\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}(\rho C_E T + \gamma T_0\mathbf{e}) = -\nabla.\left(\frac{\partial^\alpha\mathbf{q}}{\partial t^\alpha}\right) + \frac{\partial^\alpha Q}{\partial t^\alpha}, \quad 0 < \alpha \le 1. \tag{18}$$

Multiplying Eq. (18) by $\tau_0^\alpha/\alpha!$ and adding it to Eq. (13), we have

$$\frac{\partial}{\partial t}\left(1 + \frac{\tau_0^\alpha\partial^a}{\alpha!\partial t^\alpha}\right)(\rho C_E T + \gamma T_0\mathbf{e}) = -\nabla.\left(\mathbf{q} + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha\mathbf{q}}{\partial t^\alpha}\right) + Q + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha Q}{\partial t^\alpha}, \quad 0 < \alpha \le 1 \tag{19}$$

Substituting from Eq. (17), we get

$$\frac{\partial}{\partial t}\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)(\rho C_E T + \gamma T_0\mathbf{e}) = \kappa\nabla^2 T - \nabla.\Pi\boldsymbol{J} + \left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)Q, \quad 0 < \alpha \le 1 \tag{20}$$

Equation (20) is the generalized energy equation with fractional derivatives and with τ_0 taken into account. Some theories of heat-conduction law follow as limit cases for different values of the parameters α and τ_0.

Limiting cases

1. In the theory of thermoelasticity
   - Heat Eq. (20) in the limiting case $\tau_0 = 0$ transforms to the work of Biot.[53]
   - Heat Eq. (20) in the limiting case $\tau_0 = 0$ and $\upsilon = 1$ transforms to the work of Povstenko.[39]
2. In the theory of generalized thermoelasticity
   - Heat Eq. (20) in the limiting case $\alpha = 1$ transforms to the work of Lord and Shulman.[10]
   - Heat Eq. (20) in the limiting case $0 < \alpha \le 1$ transforms to the work of Sherief at el.[40]
   - Heat Eq. (20) in the limiting case $\alpha = 1$ and $\upsilon = 1$ transforms to the work of Youssef.[43]
3. In the theory of electrothermoelasticity
   - Heat Eq. (20) in the limiting case $\tau_0 = 0$ transforms to the work of Kaliski and Nowacki.[52]
   - Heat Eq. (20) in the limiting case $\alpha = 1$ transforms to the works of Ezzat and Awad,[59] Ezzat et al.,[60] and Ezzat and Atef.[61]

THE PHYSICAL PROBLEM AND STATE SPACE APPROACH

We consider a conducting thermoelectric viscoelastic solid of finite σ_0 occupying the region $x \ge 0$, where the x axis is taken to be perpendicular to the bounding plane of a half-space pointing inwards. A constant magnetic field with components $(0, H_0, 0)$, where H_i represents the magnetic field intensity, permeates the medium in the absence of an external electric field.[17] The governing equations for generalized thermoelasticity when the thermoelectric properties of the material are taken into account consist of the following:

1. The figure of merit at T_0 is

$$ZT_0 = \frac{\sigma_0 k_0^2}{\kappa} T_0 \tag{21}$$

where k_0 is the Seebeck coefficient at T_0.

2. The first Thomson relation at T_0 is

$$\pi_0 = k_0 T_0 \tag{22}$$

where π_0 is the Peltier coefficient at T_0.

3. The modified Ohm's law is

$$\mathbf{J}_i = \sigma_0 (E_i + \mu_0 \varepsilon_{ijk} \dot{u}_k H_j - k_0 T_i) \tag{23}$$

where $\mathbf{J}_i$ represents the components of the electric density vector, μ_0 is the magnetic permeability and ε_{ijk} represents the components of the strain tensor.

4. The equation of motion in the absence of body forces is

$$\sigma_{ji,j} + \mu_0 \varepsilon_{ijk} \mathbf{J}_k H_j = \rho \mathbf{u}_{i,tt} \tag{24}$$

where σ_{ij} represents the components of the stress tensor such that $\sigma_{ij} = \sigma_{ji}$, and $\mathbf{B}$ is given by $\mathbf{B}_i = \mu_0 H_i$.

5. The constitutive equation[8]

$$\mathbf{S}_{ij} = \int_0^t R(t-\tau) \frac{\partial \mathbf{e}_{ij}(x,\tau)}{\partial \tau} d\tau = \breve{R}(\mathbf{e}_{ij}), \tag{25}$$

where $\mathbf{S}_{ij}$ represents components of the stress deviator tensor.

$$\mathbf{S}_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}, \varepsilon_{ij} = \frac{1}{2}(\mathbf{u}_{i,j} + \mathbf{u}_{j,i}), \mathbf{e}_{ij} = \varepsilon_{ij} - \frac{\mathbf{e}}{3}\delta_{ij}, \mathbf{e} = \varepsilon_{kk}, \sigma = \frac{\sigma_{kk}}{3}, \tag{26}$$

where δ_{ij} is the Kronecker delta function.

$$\sigma = K_0[\mathbf{e} - 3\alpha_T (T - T_0)] \tag{27}$$

where K_0 is the bulk modulus and is equal to $\lambda + (2/3)\mu$.

$R(t)$ is relaxation function given by[62]

$$R(t) = 2\mu \left(1 - A * \int_0^t e^{-\beta^* t} t^{\alpha^* - 1} dt\right) \tag{28}$$

where α^*, β^*, and A^* are nondimensional empirical constants. $\Gamma(\alpha^*)$ is the gamma function:[7]

$$0 < \alpha^* < 1, \beta^* >, 0 \le A^* < \frac{\beta^*}{\Gamma(\alpha^*)}, R(t) > 0, \frac{d}{dt}R(t) < 0$$

Substituting from Eq. (26) into Eq. (25), we obtain

$$\sigma_{ij} = \breve{R}\left(\varepsilon_{ij} - \frac{e}{3}\delta_{ij}\right) + K_0 e \delta_{ij} - \gamma \Theta \delta_{ij}. \tag{29}$$

where Θ is a temperature equal to $T - T_0$.

6. The fractional heat equation

$$\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho C_E \frac{\partial T}{\partial t} + T_0 \gamma \frac{\partial \mathbf{e}_{ii}}{\partial t} - Q\right) = \kappa T_{,ii} - \Pi \mathbf{J}_{,i}, \quad 0 < \alpha \le 1. \tag{30}$$

In the previous equations, a comma denotes material derivatives, and the summation conventions are used. For the one-dimensional problems, all of the considered functions depend only on the space variables x and t and $\mathbf{u}$ has components $[\mathbf{u}(x,t), 0, 0]$. Because no external electric field is applied and the effect of polarization of the ionized medium can be neglected, it follows that E vanishes identically inside the medium.[17]

The components of the electromagnetic induction vector are given by

$$\mathbf{B}_x = \mathbf{B}_Z = 0, \ \mathbf{B}_y = \mu_0 H_0 = \mathbf{B}_0 \ (\text{constant})$$

whereas the components of the Lorentz force (F_i) appearing in Eq. (23) are given by

$$F_x = -\sigma \mathbf{B}_0^2 \dot{u}, F_y = F_z = 0$$

Let us introduce the following nondimensional variables:

$$x^* = c_0 \eta_0 x, \quad \mathbf{u}^* = c_0 \eta_0 \mathbf{u}, \quad t^* = c_0^2 \eta_0 t, \quad \tau_0^* = c_0^2 \eta_0 \tau_0,$$
$$\Theta^* = \frac{\gamma \Theta}{K_0}, \quad R^* = \frac{2}{3K_0} R, \quad \varepsilon = \frac{T_0 \gamma^2}{\rho^2 c_0^2 C_E}, \quad M = \frac{\sigma_0 \mathbf{B}_0^2}{K_0 \eta_0},$$
$$\sigma_{ij}^* = \frac{\sigma_{ij}}{K_0}, \quad \mathbf{q}_i^* = \frac{\gamma}{\rho c_0^3 \kappa \eta_0} q_i, \quad \eta_0 = \frac{\rho C_E}{\kappa}.$$

where c_0 is the speed of propagation of the isothermal elastic waves and is equal to the square root of K_0/ρ, $\eta_0 = \rho C_E/\kappa$, and M is the magnetic field parameter. With these dimensionless variables applied, eqs. (27)–(30) reduce to the following (with the asterisks dropped for convenience):

$$(\breve{R} + 1)\frac{\partial^2 \mathbf{u}}{\partial x^2} = \frac{\partial^2 \mathbf{u}}{\partial t^2} + M \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \Theta}{\partial x}, \tag{31}$$

$$(1 + ZT_0)\frac{\partial^2 \Theta}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\left(\Theta + \varepsilon \frac{\partial u}{\partial x}\right) - \left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right) Q, \quad 0 < \alpha \le 1 \tag{32}$$

$$\sigma = \sigma_{xx} = (\breve{R} + 1)\frac{\partial \mathbf{u}}{\partial x} - \Theta, \tag{33}$$

$$R(t) = \frac{4\mu}{3K_o}\left[1 - A^* \int_0^t e^{-\beta^* t} t^{\alpha^*-1} dt\right], \breve{R}(f(x,t)) = \int_0^t R(t-\tau)\frac{\partial f(x,\tau)}{\partial \tau} d\tau \quad (34)$$

From this point, we consider a heat source of the form $Q = Q_0\delta(x)H(t)$.

To simplify the algebra, only problems with zero initial conditions are considered. With the Laplace transform defined by the following formulas applied[63]

$$\left.\begin{aligned} L\{g(t)\} &= \bar{g}(s) = \int_0^\infty e^{-st} g(t)dt \\ L\{D^n g(t)\} &= s^n L\{g(t)\} \quad n > 0 \end{aligned}\right\}, \quad (35)$$

On both sides of eqs. (31) and (32) and with the resulting equations written in matrix form

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{Bmatrix} \bar{\Theta}(x,s) \\ \bar{\mathbf{u}}(x,s) \\ \bar{\Theta}'(x,s) \\ \bar{\mathbf{u}}'(x,s) \end{Bmatrix} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & a\varepsilon \\ 0 & \alpha s(s+M) & \alpha & 0 \end{Bmatrix}\begin{Bmatrix} \bar{\Theta}(x,s) \\ \bar{\mathbf{u}}(x,s) \\ \bar{\Theta}'(x,s) \\ \bar{\mathbf{u}}'(x,s) \end{Bmatrix} - \mathbf{Q}_0\beta\delta(x)\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \quad (36)$$

where ′ is the Laplace transform

$$L\left\{\widehat{R}\frac{\partial^2 u}{\partial x^2}\right\} = s\bar{R}\frac{\partial^2 \bar{u}}{\partial x^2}, \bar{R}(s) = \frac{4\mu}{3sK_o}\left[1 - \frac{A^*\Gamma(\alpha^*)}{(s+\beta^*)^{\alpha^*}}\right],$$
$$a = \frac{s}{1+ZT_o}\left(1 + \frac{\tau_o^\alpha}{\alpha!}s^\alpha\right),$$
$$\beta = a/s \text{ and } \omega = \frac{1}{\bar{R}s+1}$$

Equation (32) can be rewritten in constricted form as

$$\bar{\mathbf{G}}'(x,s) = \mathbf{A}(s)\bar{\mathbf{G}}(x,s) + \mathbf{B}(x,s) \quad (37)$$

where $\overline{\mathbf{G}}(x, s)$ denotes the state vector in the transform domain, whose components consist of the transformed temperature and displacement and their gradients.

To solve the system in Eq. (37), we need first to find the form of the matrix $\exp[\mathbf{A}(s)x]$.

The characteristic equation of the matrix $\mathbf{A}(s)$ has the form

$$k^4 - [\omega s(s+M) + a(1+\omega\varepsilon)]k^2 + a\omega s(s+M) = 0, \quad (38)$$

where k is a characteristic root. The Cayley–Hamilton theorem states that the matrix $\mathbf{A}$ satisfies its own characteristic equation in the matrix sense. Therefore, it follows that

$$\mathbf{A}^4 - [\omega s(s+M) + a(1+\omega\varepsilon)]\mathbf{A}^2 + a\omega s(s+M)\mathbf{I} = 0. \quad (39)$$

Equation (39) shows that $\mathbf{A}^4$ and all higher powers of $\mathbf{A}$ can be expressed in terms of $\mathbf{A}^3$, $\mathbf{A}^2$, $\mathbf{A}$, and $\mathbf{I}$, the unit matrix of order 4. The matrix exponential can now be written in the form

$$\exp[\mathbf{A}x] = a_0(x,s)\mathbf{I} + a_1(x,s)\mathbf{A}(s) + a_2(x,s)\mathbf{A}^2(s) + a_3(x,s)\mathbf{A}^3(s). \quad (40)$$

The scalar coefficients of Eq. (40) are now evaluated by the replacement of matrix $\mathbf{A}$ by its characteristic roots $\pm k_1$ and $\pm k_2$, which are the roots of the biquadratic Eq. (38) and satisfy the relations

$$k_1^2 + k_2^2 = \omega s(s+M) + a(1+\omega\varepsilon) \quad (41a)$$
$$k_1^2 k_2^2 = a\omega s(s+M) \quad (41b)$$

This leads to the following system of equations:

$$\exp(\pm k_1.x) = a_o \pm a_1 k_1 + a_2 k_1^2 \pm a_3 k_1^3 \quad (42a)$$
$$\exp(\pm k_2.x) = a_o \pm a_1 k_2 + a_2 k_2^2 \pm a_3 k_2^3 \quad (42b)$$

By solving the system of linear eqs. (42), we can determined $a_0 - a_3$ (see Appendix A).

Substituting the parameters $a_0 - a_3$ into Eq. (40), computing $\mathbf{A}^2$ and $\mathbf{A},^3$ and using eqs. (41a) and (41b), one can obtain after some lengthy algebraic manipulations the following:

$$\exp[\mathbf{A}(s).x] = L(x,s) = [\ell_{ij}(x,s)], \quad i,j = 1,2,3,4. \quad (43)$$

where the elements $\ell_{ij}(x, s)$ are given in Appendix B.

In the actual physical problem, the space is divided into two regions accordingly as $x \geq 0$ or $x < 0$; inside the region $0 \leq x < \infty$, the positive exponential terms, not bounded at infinity, must be suppressed. Thus, for $x \geq 0$, we should replace each $\sinh(kx)$ with $-\frac{1}{2}\exp(-kx)$ and each $\cosh(kx)$ with $\frac{1}{2}\exp(-kx)$. In the region $x \leq 0$, the negative exponentials are suppressed instead.

We now proceed to obtain the solution of the problem for the region $x \geq 0$. The solution for the other region is obtained by the replacement of each y with $-x$.

The formal solution of Eq. (37) can be expressed as

$$\bar{\mathbf{G}}(x,s) = \exp[\mathbf{A}(x,s)x]\left(\bar{\mathbf{G}}(0,s) + \int_0^x \exp[-\mathbf{A}(s)z]\mathbf{B}(z,s)dz\right) \quad (44)$$

Evaluating the integral in Eq. (40) using the integral properties of the Dirac δ function, we obtain

$$\bar{\mathbf{G}}(x,s) = L(x,s)x[\bar{\mathbf{G}}(0,s) + \xi(s)] \qquad (45)$$

where

$$\xi(s) = -\frac{\mathbf{Q}_0\beta}{4s}\begin{bmatrix} \frac{k_1k_2+\omega s(s+M)}{k_1+k_2} \\ 0 \\ 1 \\ \frac{\omega}{k_1+k_2} \end{bmatrix}$$

Equation (45) expresses the solution of the problem in the Laplace transform domain in terms of the vector $\xi(s)$ representing the applied heat source and the vector $\overline{\mathbf{G}}(0,s)$ representing the conditions at the plane source of heat. To evaluate the components of this vector, we note first that, because of the symmetry of the problem, the temperature is a symmetric of y, whereas the displacement is antisymmetric. It thus follows that

$$\mathbf{u}(0,t) = 0 \quad \text{or} \quad \bar{\mathbf{u}}(0,s) = 0 \qquad (46)$$

Gauss's divergence theorem is now be used to obtain the thermal condition at the plane source. We consider a short cylinder of unit base whose axis is perpendicular to the plane source of heat and whose bases lie on opposite sides of it.

Taking limits as the height of the cylinder moves toward zero and noting that there is no heat flux through the lateral surface, upon using the symmetry of the temperature field, we get

$$\mathbf{q}(0,t) = \frac{\mathbf{Q}_0}{2}H(t) \quad \text{or} \quad \bar{\mathbf{q}}(0,s) = \frac{\mathbf{Q}_0}{2s} \qquad (47)$$

With Fourier's law of heat conduction in the nondimensional form, namely

$$\bar{\mathbf{q}}(x,s) = -\frac{1}{\beta}\bar{\Theta}'(x,s) \qquad (48)$$

we obtain the condition

$$\bar{\Theta}'(0,s) = -\frac{\beta\mathbf{Q}_0}{2s} \qquad (49)$$

Equations (46) and (49) give two components of the vector $\overline{\mathbf{G}}(0,s)$. To obtain the remaining two components, we substitute $x = 0$ on both sides of Eq. (45) to obtain a system of linear equations whose solution gives

$$\bar{\Theta}'(0,s) = -\frac{\beta\mathbf{Q}_0[k_1k_2 + \omega s(s+M)]}{2s\,k_1\,k_2\,(k_1+k_2)} \qquad (50)$$

$$\bar{\mathbf{u}}'(0,s) = \frac{\beta\omega\mathbf{Q}_0}{2s(k_1+k_2)}. \qquad (51)$$

Inserting the values from eqs. (46) and (49)–(51) into the right-hand side of Eq. (45) and performing the necessary matrix operations, we obtain the temperature and the displacement component in the following form:

$$\bar{\Theta}(x,s) = -\frac{\beta\mathbf{Q}_0}{2s(k_1^2+k_2^2)}\cdot\left[\frac{k_1^2 - \omega s(s+M)}{k_1}\mathbf{e}^{\pm k_1 x} - \frac{k_2^2 - \omega s(s+M)}{k_2}\mathbf{e}^{\pm k_2 x}\right] \qquad (52)$$

$$\bar{\mathbf{u}}(x,s) = \frac{\pm\beta\omega\mathbf{Q}_0}{2s(k_1^2-k_2^2)}[\mathbf{e}^{\pm k_1 x} - \mathbf{e}^{\pm k_2 x}]. \qquad (53)$$

Substituting from eqs. (52) and (53) into Eq. (33), we get the stress component in the form

$$\bar{\sigma}(x,s) = \frac{\beta\omega(s+M)\mathbf{Q}_0}{2k_1k_2(k_1^2-k_2^2)}[k_2\mathbf{e}^{\pm k_1 x} - k_1\mathbf{e}^{\pm k_2 x}]. \qquad (54)$$

We can obtain the strain component from Eq. (33) as

$$\bar{\mathbf{e}}(x,s) = \frac{\beta\omega\mathbf{Q}_0}{2s(k_1^2-k_2^2)}[k_1\mathbf{e}^{\pm k_1 x} - k_1\mathbf{e}^{\pm k_2 x}]. \qquad (55)$$

In the previous equations, the upper (plus) sign indicates the solution in the region $x < 0$, whereas the lower (minus) sign indicates the region $x \geq 0$, respectively.

Those complete the solution in the Laplace transform domain.

INVERSION OF THE LAPLACE TRANSFORM

We now outline the numerical inversion method used to find the solution in the physical domain. This numerical technique has the advantages that it is easy to implement (relatively speaking), gives good results, and converges quickly. Let $\bar{g}(x,s)$ be the Laplace transform of a function $g(x,t)$. The inversion formula for Laplace transforms can be written as follows:

$$g(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathbf{e}^{st}\bar{g}(x,s)ds, \qquad (56)$$

where c is an arbitrary constant greater than all of the real parts of the singularities of $\bar{g}(x,s)$.

With $s = c + iy$, the previous integral takes the form

$$g(x,t) = \frac{\mathbf{e}^{ct}}{2\pi}\int_{-\infty}^{\infty}\mathbf{e}^{ity}\bar{g}(x,c+iy)dy. \qquad (57)$$

TABLE I
Values of the Constants

$\kappa = 386$ N/Ks
$\alpha_T = 1.78 \times 10^{-5}$ K^{-1}
$C_E = 383.1$ m^2 s^{-2} K^{-1}
$\eta_0 = 8886.73$ s m^{-2}
$\mu = 3.86 \times 10^{10}$ N m^{-2}
$\lambda = 7.76 \times 10^{10}$ N m^{-2}
$\rho = 8954$ kg m^{-3}
$K_0 = 10.33 \times 10^{10}$ N m^{-2}
$c_0 = 3397.1$ m s^{-1}
$T_0 = 293$ K
$\varepsilon = 0.0168$
$\mu_0 = 1.256 \times 10^{-6}$ Ns^2 C^{-2}
$\tau_0 = 0.02$
$B_0 = \mu_0 H_0 = 1$ Tesla
$\alpha^* = 0.5$
$\beta^* = 0.05$
$A^* = 0.106$

Expanding the function $h(x,t) = \exp(-ct)g(x,t)$ in a Fourier series in the interval [0, 2L], we obtain the approximate formula proposed by Honig and Hirdes:[51]

$$g(x,t) = g_\infty(x,t) + E_D, \tag{58}$$

where E_D is the discretization error and

$$g_\infty(x,t) = \frac{1}{2}c_0(x,t) + \sum_{k=1}^{\infty} c_k(x,t) \quad \text{for } 0 \le t \le 2L, \tag{59}$$

and

$$c_k(x,t) = \frac{\mathbf{e}^{ct}}{L}\mathrm{Re}[\mathbf{e}^{ik\pi t/L}\bar{g}(x, c + ik\pi t/L)], \quad k = 0,1,2,\ldots \quad \text{for } 0 \le t \le 2L \tag{60}$$

where E_D can be made arbitrarily small if c is large enough.[51]

Because the infinite series in Eq. (59) can be summed up to a finite number (N) of terms, the approximate value of $g(x,t)$ becomes

$$g_N(x,t) = \frac{1}{2}c_0(x,t) + \sum_{k=1}^{N} c_k(x,t) \quad \text{for} \quad 0 \le t \le 2L \tag{61}$$

Using the above formula to evaluate $g(x,t)$, we introduce a truncation error (E_T) that must be added to E_D to produce the total approximation error.

Two methods are used to reduce the total error. First, the Korrecktur method[51] is used to reduce E_D. Next, the ε algorithm is used to reduce E_T and, hence, to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function $g(x,t)$:

$$g(x,t) = g_\infty(x,t) - e^{2cL}g_\infty(x, 2L+t) + t + E'_D$$

where $|E'_D| \ll |E_D|$. Thus, the approximate value of $g(x,t)$ becomes

$$g_{NK}(x,t) = g_N(x,t) - e^{-2cL}g_{N'}(x, 2L+t) \tag{62}$$

where N' is an integer such that $N' < N$.

We now describe the ε algorithm that is used to accelerate the convergence of the series in Eq. (61). Let N be an odd natural number, and let the following be the sequence of partial sums of a series in Eq. (61):

$$s_m(x,t) = \sum_{k=1}^{m} c_k(x,t)$$

We define the ε sequence by $\varepsilon_{0,m} = 0$, $\varepsilon_{1,m} = s_m$, and $\varepsilon_{p+1,m+1} = \varepsilon_{p-1,m+1} + 1/(\varepsilon_{p,m+1} - \varepsilon_{p,m})$, $p = 1, 2, 3, \ldots$.

It can be shown that the sequence[51] $\varepsilon_{1,1}$, $\varepsilon_{3,1}$, $\varepsilon_{5,1}$, $\ldots$ $\varepsilon_{N,1}$ converges to $g(x,t) + E_D - c_0/2$ faster than the sequence of partial sums s_m, $m = 1, 2, 3, \ldots$.

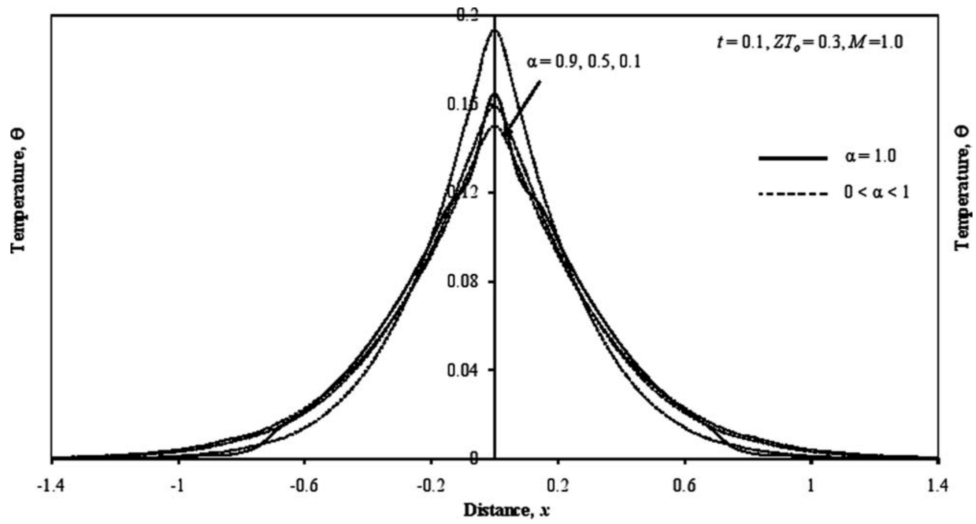

Figure 1 Dependence of the temperature on the distance for different values of α.

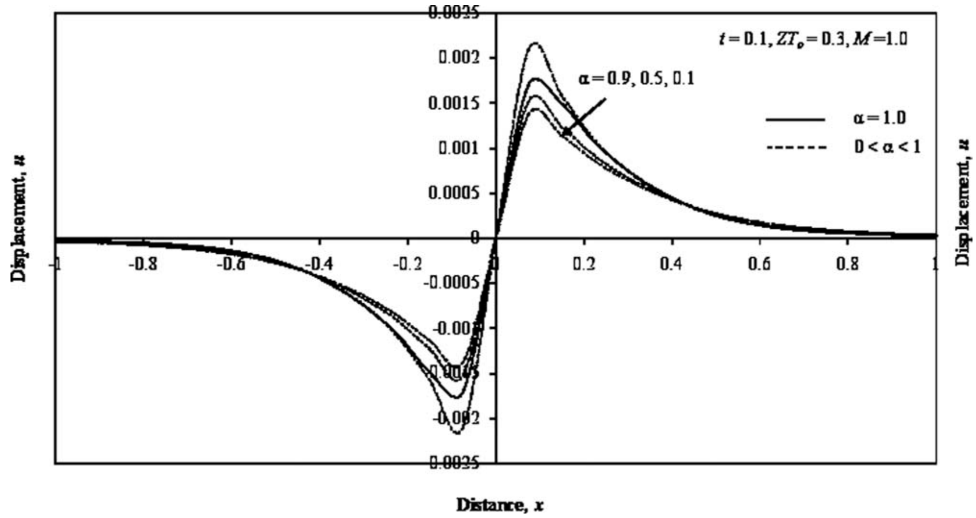

Figure 2 Dependence of the displacement on the distance for different values of α.

The actual procedure used to invert the Laplace transform consists of the use of Eq. (62) together with the ε algorithm. The values of c and L are chosen according to the criteria outlined in Honig and Hirdes.[51]

NUMERICAL RESULTS AND DISCUSSION

Copper material was chosen for purposes of numerical evaluations. The constants of the problem used are shown in Table I.[64]

The investigation of the effect of α on the thermoelectric material with heat source distribution in the presence of a magnetic field was carried out in the preceding sections. The computations were performed for a value of time, namely, $t = 0.1$. The numerical technique outlined previously was used to obtain temperature, displacement, stress, and strain. The results are represented graphically at different positions of x in Figures 1–8. In these figures, we noticed the difference in all functions for the value of α ($0 < \alpha \leq 1$), where the case of $\alpha = 1$ (normal conductivity) indicates the old situation and the case $0 < \alpha < 1$ (weak conductivity) indicates the new theory. For a normal conductivity, $\alpha = 1$, the results coincided with all of the previous results of applications that were taken in the context of generalized thermoelasticity in the absence of the effects of thermoelectric properties on the various fields.

In Figures 1–4, which exhibit the space variation of the temperature, displacement, strain, and stress fields at different values of α, we observed the following:

- The fields were continuous functions for different values of α ($0 < \alpha < 1$).

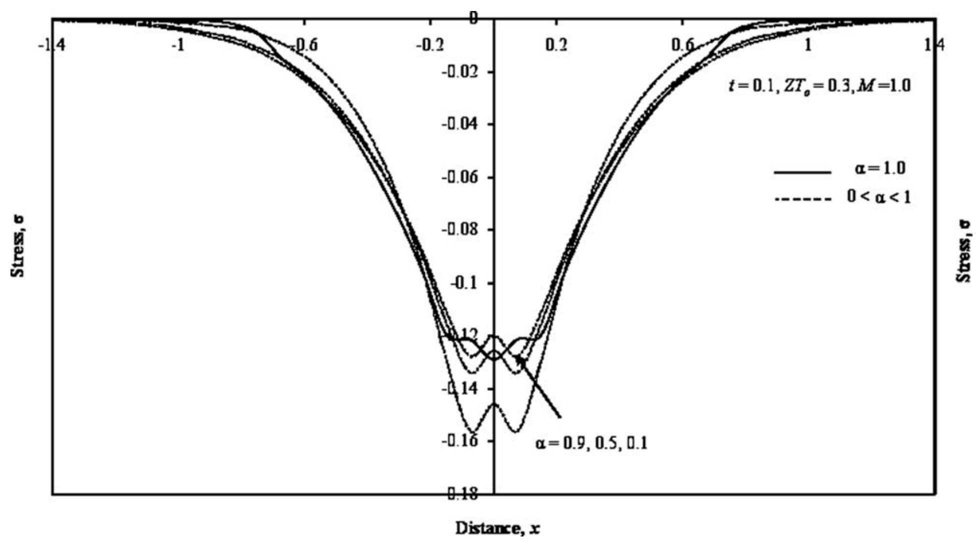

Figure 3 Dependence of the stress on the distance for different values of α.

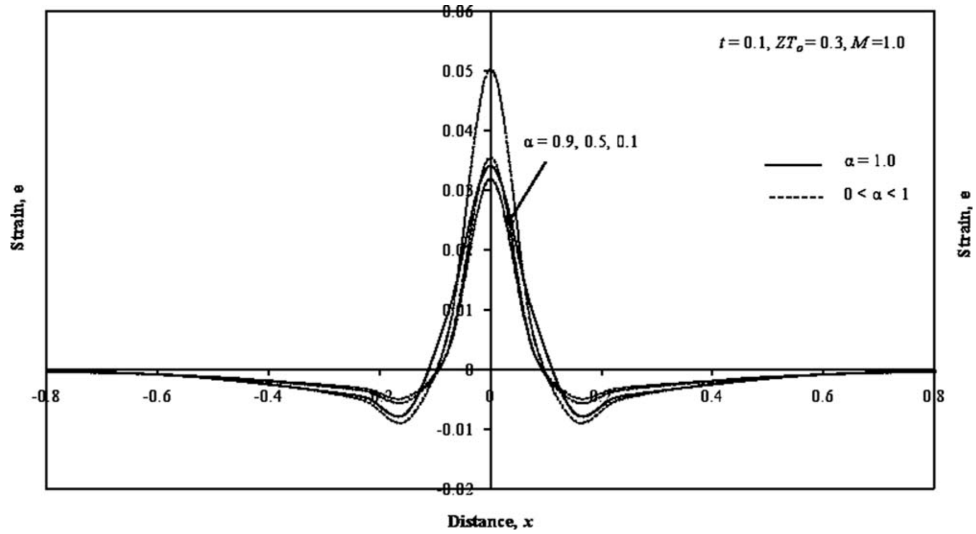

Figure 4 Dependence of the temperature on the distance for different values of α.

- The fractional order α had a significant effect on the fields.
- The waves reached a steady state depending on the value of the fractional orders α.
- The curves were smoother in the case where $0 < \alpha < 1$.
- The waves cut the x axis more rapidly when $\alpha = 1$ than when $0 < \alpha < 1$.
- Figures 1 and 3 display the temperature and stress distributions for the wide range $0 \leq x \leq 1.4$ at a value of time t of 1.0 and for different values of the differential α ($0 < \alpha < 1$). We noticed that for a wide range of $0 \leq x < 0.3$, the increasing value of the parameter α caused decreases in the temperature and magnitude of stress, whereas through the interval $0.3 \leq x \leq 1.4$, the decreasing value of the parameter α caused decreases in the temperature and magnitude of stress.
- In Figures 2 and 4, the displacement and the strain fields show the same behavior as the temperature and stress fields, except in the wide range of x.
- Figures 5–8 show the space variation of the temperature, displacement, strain, and stress fields at different values of ZT_0 (1.0, 0.5, and 0.1). The important phenomenon observed in these figures was that the effects of ZT_0 on the entire fields were identically the same as the effects of the introduced fractional order on the corresponding fields.

CONCLUDING REMARKS

- The main goal of this work was to introduce a new mathematical model for the Fourier law of

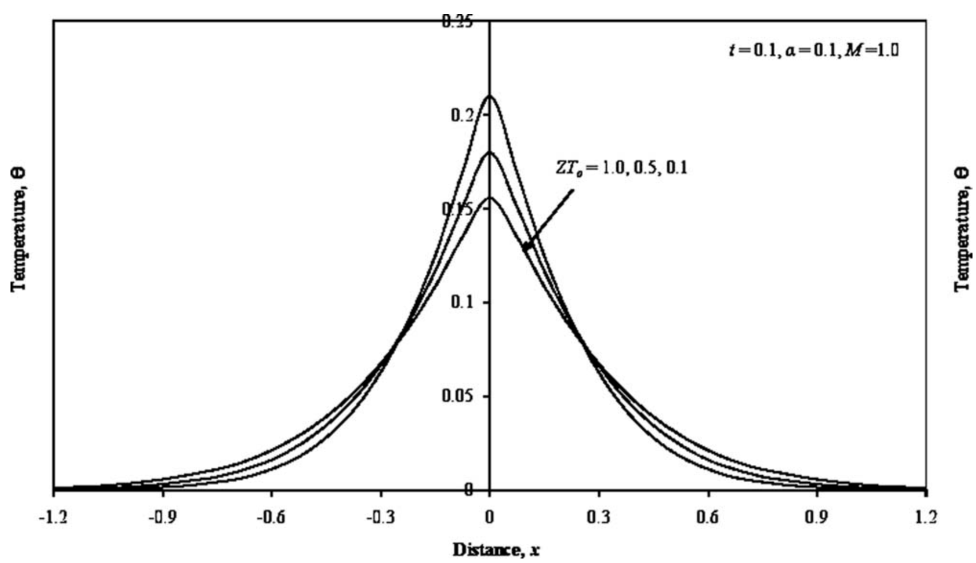

Figure 5 Dependence of the temperature on the distance for different values of ZT_0.

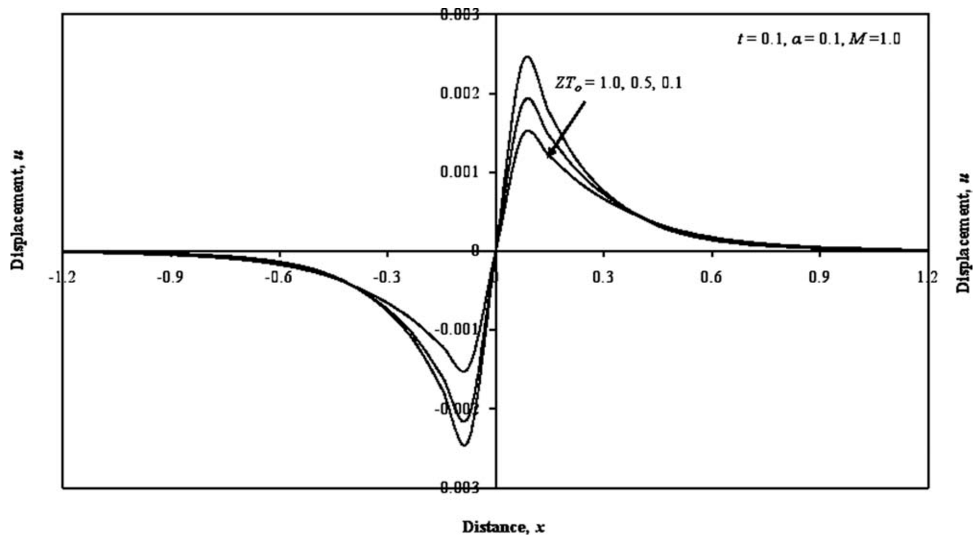

Figure 6 Dependence of the displacement on the distance for different values of ZT_0.

heat conduction with time-fractional order and to include the thermoelectric ZT_0. This model will enable us to improve the efficiency of a thermoelectric material ZT_0. It is known that to achieve a high thermoelectric material ZT_0, one requires a low thermal conductivity.[46] This can occur for small values of α.

- Previously, the discontinuity of the stress distribution was a critical situation, and no one has explained the reason physically, whereas in the context of the new theory of thermoviscoelasticity with fractional order heat transfer, the stress function is continuous.
- This article indicates that the generalized theory of thermoelectric viscoelasticity of fractional order heat transfer describes the behavior of the particles of an elastic body more realistically than the theory of generalized thermoelasticity with integer order.
- According to this new theory, we have to construct a new classification for materials according to their fractional parameter α, where this parameter becomes a new indicator of a material's ability to conduct heat under the effect of thermoelectric properties.
- The results provide a motivation to investigate conducting thermoelectric materials as a new class of applicable thermoelectric viscoelastic materials; these materials include beryllium, magnesium, calcium, barium, and steel.
- To the best of our knowledge, the consequent thermoelectric ZT_0 of all of the iodine-doped copolymer with the stretch treatment shows one of the best thermoelectric performances among conducting polymers ever reported; this is comparable with that of an inorganic thermoelectric material, such as β-$FeSi_2$, as reported by Hiroshige et al.[65]

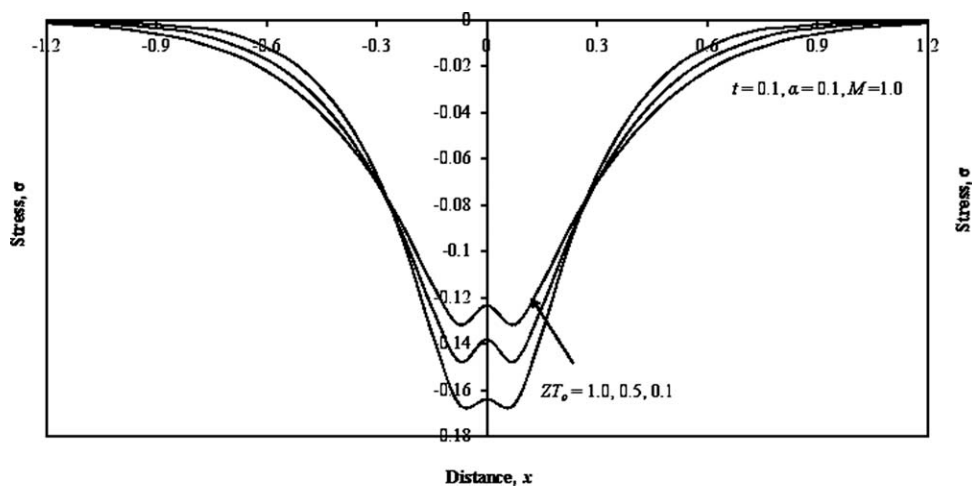

Figure 7 Dependence of the stress on the distance for different values of ZT_0.

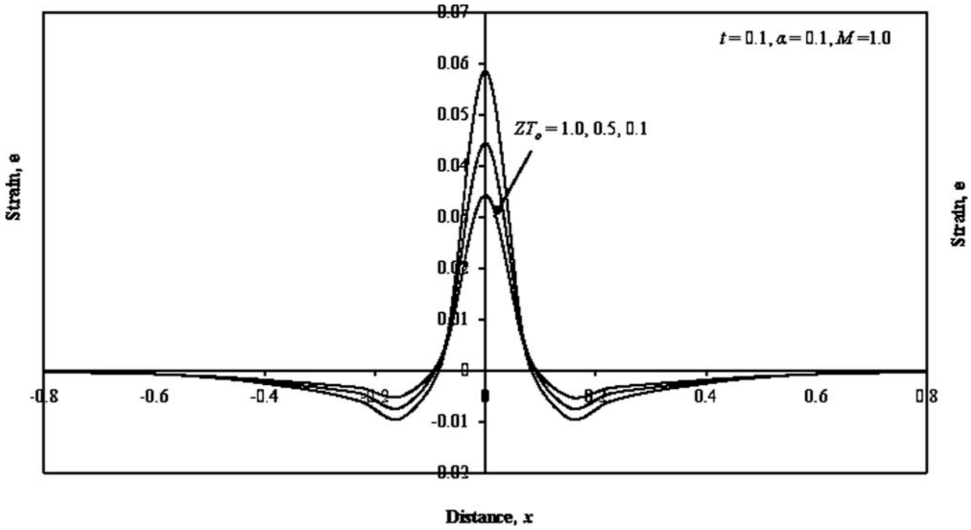

Figure 8 Dependence of the strain on the distance for different values of ZT_0.

APPENDIX A

$$a_0 = \frac{k_1^2\cosh k_2x - k_2^2\cosh k_1x}{k_1^2-k_2^2}, a_1 = \frac{k_1^3\sinh k_2x - k_2^3\sinh k_1x}{k_1k_2\left(k_1^2-k_2^2\right)}$$

$$a_2 = \frac{\cosh k_1x - \cosh k_2x}{k_1^2-k_2^2}, a_3 = \frac{k_2\sinh k_1x - k_1\sinh k_2x}{k_1k_2\left(k_1^2-k_2^2\right)}$$

APPENDIX B

$$\ell_{11} = \frac{\left(k_1^2-a\right)\cosh k_2x - \left(k_2^2-a\right)\cosh k_1x}{k_1^2-k_2^2}, \ell_{12} = a\varepsilon\omega s(s+M)\left[\frac{k_2\sinh k_1x - k_1\sinh k_2x}{k_1k_2\left(k_1^2-k_2^2\right)}\right]$$

$$\ell_{13} = \frac{k_2[k_1^2-\omega s(s+M)]\sinh k_1x - k_1[k_2^2-\omega s(s+M)]\sinh k_2x}{k_1k_2\left(k_1^2-k_2^2\right)}$$

$$\ell_{14} = a\varepsilon\left[\frac{\cosh k_1x - \cosh k_2x}{k_1^2-k_2^2}\right], \ell_{21} = a\left[\frac{k_2\sinh k_1x - k_1\sinh k_2x}{k_1k_2\left(k_1^2-k_2^2\right)}\right]$$

$$\ell_{22} = \frac{[k_1^2-\omega s(s+M)]\cosh k_2x - [k_2^2-\omega s(s+M)]\cosh k_1x}{k_1^2-k_2^2}$$

$$\ell_{23} = \frac{\cosh k_1x - \cosh k_2x}{k_1^2-k_2^2}, \ell_{24} = \frac{k_2(k_1^2-a)\sinh k_1x - k_1(k_2^2-a)\sinh k_2x}{k_1k_2\left(k_1^2-k_2^2\right)}$$

$$\ell_{31} = a\left\{\frac{k_2[k_1^2-\omega s(s+M)]\sinh k_1x - k_1[k_2^2-\omega s)(s+M)]\sinh k_2x}{k_1k_2\left(k_1^2-k_2^2\right)}\right\}$$

$$\ell_{32} = a\varepsilon\omega s(s+M)\left[\frac{\cosh k_1x - \cosh k_2x}{k_1^2-k_2^2}\right], \ell_{33} = \frac{[k_1^2-\omega s(s+M)]\cosh k_1x - [k_2^2-\omega s(s+M)]\cosh k_2x}{k_1^2-k_2^2}$$

$$\ell_{34} = a\varepsilon\left[\frac{k_1\sinh k_1x - k_2\sinh k_2x}{k_1^2-k_2^2}\right],$$

$$\ell_{41} = a\left[\frac{\cosh k_1x - \cosh k_2x}{k_1^2-k_2^2}\right],$$

$$\ell_{42} = \omega s(s+M)\left[\frac{k_2(k_1^2-a)\sinh k_1x - k_1(k_2^2-a)\sinh k_2x}{k_1k_2(k_1^2-k_2^2)}\right]$$

$$\ell_{43} = \frac{k_1\sinh k_1x - k_2\sinh k_2x}{k_1^2-k_2^2}, \ell_{44} = \frac{(k_1^2-a)\cosh k_1x - (k_2^2-a)\cosh k_2x}{k_1^2-k_2^2}$$

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